MTH 166

Lecture-2

Equations Reducible to Exact Form-I

Topic:

Equations Reducible to Exact Form

Learning Outcomes:

1. What is a Non-Exact Differential Equations.

2. What is an Integrating Factor (I.F.).

3. How to find an I.F. using different methods (Two methods).

4. How to convert a Non-exact equation into an Exact equation using I.F. and find its solution.

Non-Exact Differential Equation:

Let us consider:

$$\boldsymbol{xdy - ydx + a\left(x^2 + y^2\right)dx = 0} \qquad (1)$$

Equation (1) can be re-written as:

$$(ax^2 + ay^2 - y)dx + xdy = 0$$

Comparing it with: $Mdx + Ndy = 0$
Here $M = (ax^2 + ay^2 - y) \Rightarrow \frac{\partial M}{\partial y} = 2ay - 1$
And $N = x \Rightarrow \frac{\partial N}{\partial y} = 1$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ **,** So, Equation (1) is Non-Exact differential equation.

Integrating Factor (I.F.):

An Integrating Factor (I.F.) is a factor which when multiplied with a non-exact differential equation, makes it an exact differential equation.

Means: $(Non - Exact\ Equ.) \times I.F. = Exact\ Equ.$

**There can be more than one I.F. for one non-exact diff. equ.

Methods to find Integrating Factor (I.F.):

1. Inspection Method:

In this method we look for a particular expression: $(xdy - ydx)$ which is non-exact.

There are many I.F. available to make this expression an exact one, like;

$\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}, \frac{1}{x^2-y^2}.$

We know: $(Non - Exact\ Equ.) \times I.F. = Exact\ Equ.$

1. $(xdy - ydx) \times \frac{1}{x^2} = \frac{(xdy-ydx)}{x^2} = d\left(\frac{y}{x}\right)$ (Quotient rule)
2. $(xdy - ydx) \times \frac{1}{y^2} = \frac{-(ydx-xdy)}{y^2} = -d\left(\frac{x}{y}\right)$ (Quotient rule)
3. $(xdy - ydx) \times \frac{1}{xy} = \frac{(xdy-ydx)}{xy} = d\left[\log\left(\frac{y}{x}\right)\right]$
4. $(xdy - ydx) \times \frac{1}{x^2+y^2} = \frac{(xdy-ydx)}{x^2+y^2} = d\left[tan^{-1}\left(\frac{y}{x}\right)\right]$
5. $(xdy - ydx) \times \frac{1}{x^2-y^2} = \frac{(xdy-ydx)}{x^2-y^2} = d\left[\frac{1}{2}\log\left(\frac{x+y}{x-y}\right)\right]$

<u>Solve the following differential equations:</u>

<u>Problem 1:</u> $\boldsymbol{xdy - ydx + a(x^2 + y^2)dx = 0}$

<u>Solution:</u> $(xdy - ydx) + a(x^2 + y^2)dx = 0$ (1)

Multiplying both sides by I.F. = $\frac{1}{x^2+y^2}$

$$[(xdy - ydx) + a(x^2 + y^2)dx] \times \frac{1}{x^2+y^2} = 0$$

$$\Rightarrow \frac{(xdy-ydx)}{x^2+y^2} + adx = 0$$

$$\Rightarrow d\left[tan^{-1}\left(\frac{y}{x}\right)\right] + adx = 0$$

Integrating both sides:

$$\int d\left[tan^{-1}\left(\frac{y}{x}\right)\right] + a\int dx = c$$

$\Rightarrow tan^{-1}\left(\frac{y}{x}\right) + ax = c$ **Answer.**

Solve the following differential equations:

Problem 2: $\boldsymbol{ydx - xdy + 3x^2y^2e^{x^3}dx = 0}$

Solution: $(ydx - xdy) + 3x^2y^2e^{x^3}dx = 0$ (1)

Multiplying both sides by I.F. = $\frac{1}{y^2}$

$$\left[(ydx - xdy) + 3x^2y^2e^{x^3}dx\right] \times \frac{1}{y^2} = 0$$

$$\Rightarrow \frac{(ydx - xdy)}{y^2} + 3x^2e^{x^3}dx = 0$$

$$\Rightarrow d\left(\frac{x}{y}\right) + 3x^2e^{x^3}dx = 0$$

Integrating both sides:

$$\int d\left(\frac{x}{y}\right) + \int e^{x^3}.3x^2dx = c \qquad \left[\because \int e^{f(x)}.f'(x)dx = e^{f(x)} + c\right]$$

$$\Rightarrow \left(\frac{y}{x}\right) + ax = c$$ **Answer.**

Methods to find Integrating Factor (I.F.):

2. If $Mdx + Ndy = 0$ is a homogeneous differential equation in x and y, then:

I.F. $= \frac{1}{Mx+Ny}$

Solve the following differential equations:

Problem 1. $\boldsymbol{(x^3 + y^3)dx - xy^2dy = 0}$

Solution: $(x^3 + y^3)dx - xy^2dy = 0$ (1)

Comparing it with: $Mdx + Ndy = 0$

Here $M = (x^3 + y^3) \Rightarrow \frac{\partial M}{\partial y} = 3y^2$

And$N = -xy^2 \Rightarrow \frac{\partial N}{\partial x} = -y^2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .

We need to make it exact using I.F.

Look at equation: $(x^3 + y^3)dx - xy^2 dy = 0$ (1)

Which is homogeneous in x and y

★ An equation is said to be homogeneous is each term od M and N is of same degree.

Here $M = (x^3 + y^3)$. Each of these two terms x and y have same degree 3.

And $N = -x^1 y^2$ is also of same degree 3 as sum of powers of x and y is 3

Thus, since equation (1) is homogeneous of degree 3.

So, I.F. = $\frac{1}{Mx+Ny}$

⇒I.F.= $\frac{1}{(x^3+y^3)x+(-xy^2)y} = \frac{1}{x^4}$

Multiplying equation (1) by I.F. to convert it to an exact equation:

$$\left[(x^3 + y^3)dx - xy^2dy\right] \times \frac{1}{x^4} = 0$$

$$\Rightarrow \left(\frac{1}{x} + \frac{y^3}{x^4}\right) dx - \frac{y^2}{x^3} dy = 0 \qquad (2)$$

Which is an exact differential equation.

Solution: $\int_{y=con.} Mdx + \int (Terms\ of\ N\ free\ from\ x)dy = C$

$$\Rightarrow \int_{y=con.} \left(\frac{1}{x} + \frac{y^3}{x^4}\right) dx + 0 = c$$

$\Rightarrow \log x - \frac{y^3}{3x^3} = c$ **Answer.**

Problem 2. $\boldsymbol{(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0}$

Solution: $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$ (1)

Comparing it with: $Mdx + Ndy = 0$

Here $M = (x^2y - 2xy^2) \Rightarrow \frac{\partial M}{\partial y} = x^2 - 4xy$

And $N = (3x^2y - x^3) \Rightarrow \frac{\partial N}{\partial x} = 6xy - 3x^2$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, So, Equation (1) is Non-exact .

We need to make it exact using I.F.

since equation (1) is homogeneous of degree 3.

So, I.F. = $\frac{1}{Mx+Ny}$

$\Rightarrow$I.F.$= \frac{1}{(x^2y-2xy^2)x+(3x^2y-x^3)y} = \frac{1}{x^2y^2}$

Multiplying equation (1) by I.F. to convert it to an exact equation:

$$[(x^2y-2xy^2)dx+(3x^2y-x^3)dy]\times\frac{1}{x^2y^2}=0$$

$$\Rightarrow\left(\frac{1}{y}-\frac{2}{x}\right)dx+\left(\frac{3}{y}-\frac{x}{y^2}\right)dy=0 \qquad (2)$$

Which is an exact differential equation.

Solution: $\int_{y=con.} Mdx+\int(Terms\ of\ N\ free\ from\ x)dy=C$

$$\Rightarrow\int_{y=con.}\left(\frac{1}{y}-\frac{2}{x}\right)dx+\int\frac{3}{y}dy=c$$

$\Rightarrow\frac{x}{y}-2\log x+3\log y=c$ **Answer.**